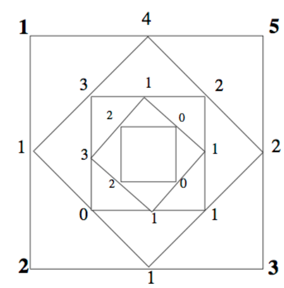
Math Fest problems (Topic: “Geometry in Math Olympiads”)

Notes

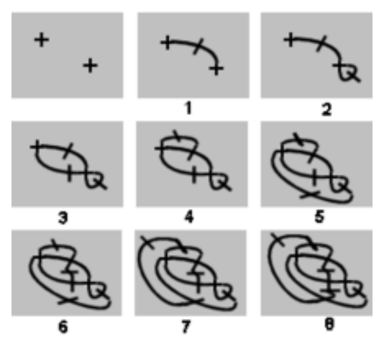
- Questions have a * rating by level of difficulty/age. This is only meant as a guide.

---

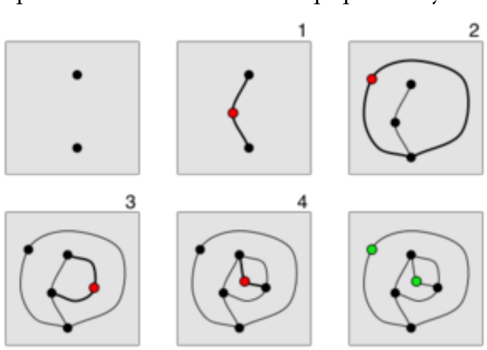
1. *The edges of K6, the complete graph with 6 vertices, are each colored in red or blue. Prove that there is a monochromatic triangle.
2. *Let G be a finite simple graph, and there is a light bulb at each vertex of G. Initially, all the lights are off. Each step we are allowed to chose a vertex and toggle the light at that vertex as well as all its neighbors’. Show that we can get all the lights to be on at the same time.
3. *The four numbers game begins with four nonnegative integers, a,b,c,d and a square (see right). One number is placed at each corner of the square. The first step is then obtained by creating a new square inside the original square with corners at the midpoints of the sides of the previous square. Each new corner is labeled with the absolute value of the difference between the two neighboring labels. This process continues until the difference between the numbers at each of the vertices is zero. The game is over when all four labels are zeros. Suppose we use an n-gon instead of a square. For which n is the game sure to end? What about 3D shapes? (See Circle of Differences handout for exploratory questions.)



4. *Brussels Sprouts: This game starts with a number of crosses, i.e. spots with four free ends. Each move involves joining two free ends with a curve, again not crossing any existing line, and then putting a short stroke across the line to create two new free ends.

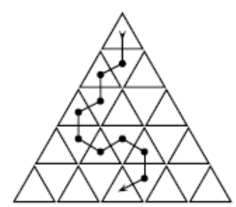


5. **Sprouts: The game is played by two players, starting with a few spots drawn on a sheet of paper. Players take turns, where each turn consists of drawing a line between two spots (or from a spot to itself) and adding a new spot somewhere along the line. The players are constrained by the following rules: The line may be straight or curved, but must not touch or cross itself or any other line. The new spot cannot be placed on top of one of the endpoints of the new line. Thus the new spot splits the line into two shorter lines. No spot may have more than three lines attached to it. For the purposes of



this rule, a line from the spot to itself counts as two attached lines and new spots are counted as having two lines already attached to them. In so-called *normal play*, the player who makes the last move wins.

6. **Chomp is a mathematical game of strategy in which two players take turns removing vertices and edges from graphs. Players move in turn rather than simultaneously and each player has complete information about the state of the game while making a move. The winner of the game is the one who removes the last vertex, leaving the loser with nothing to remove.
7. **Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that the player who signs first will always win by playing as well as possible.
8. **7 students take a mathematical exam. Every problem was solved by at most 3 students. For every pair of students, there is at least one problem that they both solved. Determine, with proof, the minimum number of problems on this exam.
9. ***A closed unit disc contains 7 points such that any two of them are at least unit distance apart. Show that the center of the disc is one of the 7 points.
10. ***Each vertex of a regular polygon is colored with one of a finite number of colors so that the points of the same color are the vertices of some new regular polygon. Prove that at least two of the polygons obtained are congruent.
11. ***A square grid on the Euclidean plane consists of all point (m,n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?
12. ***Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let f(n) be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for n=5. Determine the value of f(n).



13. ***Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a non-negative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.
14. ***What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?
15. ***What is the maximum number of rational points that can lie on a circle in R^2 whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)
16. ***You are in a boat in the center of a circular lake. There is a rabid dog on the shore that really wants to eat you. If you can get to shore and the dog is not there waiting, then you can outrun him to your car. If the dog can run four times as fast as you can row, how can you escape to your car? Assume that neither the dog nor you can swim and that the dog never sleeps and is smarter than you.
17. ****Suppose that S is a finite set of points in the plane such that the area of triangle ΔABC is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.
18. ****Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.
19. ****Let Q be a cube of side-length 1 foot, suspended, in some orientation, above a flat earth, and illuminated by a sun whose rays are purely vertical. Let A be the area of the shadow of Q in square feet, and let h be the height of Q in feet. Show that A = h. Can this be generalized to other regular polyhedra?

20. ****Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?
21. *****The 30 edges of an icosahedron are distinguished by labeling them 1, 2, . . . , 30. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?
22. *****A triangulation T of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in T. Say that T is admissible if every internal vertex is shared by 6 or more triangles. Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.